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AUTHOR(S):

Saito, Seiji; Ishii, Hiroaki

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Minimax Theorems of Convexlike Functions

大阪大学大学院情報科学研究科情報数理学専攻 齋藤誠慈 (Seiji SAITO)

大阪大学大学院情報科学研究科情報数理学専攻 石井博昭 (Hiroaki ISHII)

Graduate School of Information Science and Technology, Osaka University

E-mail: saito@ist.osaka-u.ac.jp

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1 Introduction

In this study we consider fuzzy numbers with bounded supports due to [3] and we treat some type of fuzzy optimization problems, which arise from linear optimization problems and are analyzed under assumptions of the fuzzy goal and fuzzy constraints of decision makers. [5] gives an existence criterion for optimal solutions of the fuzzy optimization problems. In Section 2 the existence of optimal solutions means that there exists at least one solution for systems of inequalities concerning concave functions by applying Ky Fan's theorem. In Section 3 we show an extension of Ky Fan's theorem, in which functions are not convex but quasiconvex. In proving the extension we apply fixed point theorems for set-valued mappings. In Section 4 we deal with definitions of convexlike or concavelike functions in the similar way to Chapter 6 in [7] as well as we get minimax theorems under conditions that functions of two variables are lower semi-continuous and quasiconvex in one variable and concavelike in the other.

2 Existence Criterion

Let us denote by \mathbf{R} the set of real numbers and $I = [0, 1]$. In [3] the set of fuzzy numbers is characterized by membership functions as follows:

Definition 1 Let $\mathcal{F}(\mathbf{R})$ be the set of fuzzy numbers $u : \mathbf{R} \rightarrow I$ satisfying the following conditions (i) - (iii) (see [3]):

- (i) $u(\cdot)$ is upper semi-continuous on \mathbf{R} ;
- (ii) the α -cut set $L_\alpha(u) = \{y \in \mathbf{R} : u(y) \geq \alpha\}$ is bounded for $\alpha > 0$ and $L_0(u) = \overline{\cup_{0 < \alpha \leq 1} L_\alpha(u)}$ is bounded ;
- (iii) $u(\cdot)$ is fuzzy convex, i.e.,

$$u(\lambda y_1 + (1 - \lambda)y_2) \geq \min[u(y_1), u(y_2)]$$
for $y_i \in \mathbf{R}, i = 1, 2$ and $\lambda \in \mathbf{R}$ with $0 \leq \lambda \leq 1$;
- (iv) there exists one and only one $m \in \mathbf{R}$ such that $u(m) = 1$.

The α -cut set $L_\alpha(u)$ is compact in \mathbf{R} for each $\alpha \in I$ from the above Conditions (i) and (ii), since (i) means that $L_\alpha(u)$ is closed for $\alpha \in I$.

Remark 1 Under the above Conditions (ii) and (iv) the following statements (a)-(d) concerning the the function $u : \mathbf{R} \rightarrow I$ are equivalent each other:

- (a) $u(\cdot)$ is fuzzy convex;
- (b) $L_\alpha(u)$ is convex for any $\alpha \in I$;
- (c) $u(\cdot)$ is non-decreasing on $(-\infty, m]$ and that $u(\cdot)$ is non-increasing on $[m, \infty)$;
- (d) $L_\alpha(u) \subset L_\beta(u)$ for $\alpha > \beta$.

From (a) it is clear that (b) holds. If we suppose that (a) doesn't hold but (b) hold, this leads to a contradiction. It can be seen that (c) leads to (d) and the converse holds. Suppose that for any $m_1 \in \mathbf{R}$ with $m_1 > m$ there exist $y_1 < y_2 \leq m_1$ such that $u(y_1) > u(y_2)$ under Condition (ii) and (a). Then it leads to a contradiction. From (c), it follows that (a) holds.

In the following definition we give the quasiconvexity of functions.

Definition 2 Let C be a convex set in a linear space and f a mapping from C to \mathbf{R} . It is said that f is quasiconcave if $f(\lambda y_1 + (1 - \lambda)y_2) \geq \min[f(y_1), f(y_2)]$ for $y_i \in C, i = 1, 2$ and $0 \leq \lambda \leq 1$. It is said that f is quasiconvex if

$$f(\lambda y_1 + (1 - \lambda)y_2) \leq \max[f(y_1), f(y_2)]$$

for $y_i \in C, i = 1, 2$ and $0 \leq \lambda \leq 1$.

Remark 2 In the same way as in Remark 2.1 it is easily seen that $f : C \rightarrow \mathbf{R}$ is quasiconvex if and only if the lower level set $L(f; \gamma) = \{x \in C : f(x) \leq \gamma\}$ is convex for any $\gamma \in \mathbf{R}$.

Next we consider the following linear optimization problem (e.g. [4]):

$$a_0^T x \preceq b_0 \quad \text{subject to} \quad a_i^T x \preceq b_i, \quad (2.1)$$

$$i = 1, 2, \dots, m, \quad x \geq 0, \quad (2.2)$$

where the symbol “ \preceq ” denotes a relaxed or fuzzy version of the ordinary inequality “ \leq ”. The first fuzzy inequality (fuzzy goal) means that “the objective function $a_0^T x$ should be essentially smaller than or equal to an aspiration level $b_0 \in \mathbf{R}$ of the decision maker (DM)” and the second (fuzzy constraints of DM) means that “the constraints $a_i^T x$ should be essentially smaller than or equal to $b_i \in \mathbf{R}, i = 1, \dots, m$ ”. Membership functions $u_i \in \mathcal{F}(\mathbf{R}), i = 0, 1, \dots, m$, and it follows that $u_i(y)$ is non-decreasing in $y \in [C_i, b_i]$, non-increasing in $y \in [b_i, D_i]$ and $u_i(y) \equiv 0$ elsewhere. Here $C_i \leq b_i \leq D_i$ are constants. Let u_i be concave on the set $[C_i, D_i]$. Put $S_i = \{x \in \mathbf{R}^n : C_i \leq a_i^T x \leq D_i\}$ and $S = \cap_{i=0}^n S_i$.

Then, in order to solve the above problem, we have the following optimization problem:

$$\text{maximize } u(x), \quad (2.3)$$

$$\text{where } u(x) = \min_{0 \leq i \leq m} [u_i(a_i^T x)]. \quad (2.4)$$

In [5] we showed the existence criterion for optimal solutions of fuzzy optimization problems as follows:

Theorem 1 Let $u_i(\cdot) \in \mathcal{F}$ for $i = 0, 1, \dots, m$. The following statements (i) and (ii) hold;

- (i) Let $\mu_0 = \max_x \min_i u_i(a_i^T x)$. Then we have

$$\begin{aligned} \mu_0 &= \max\{0 < \alpha \leq 1 : \cap_{i=0}^m L_\alpha(u_i) \neq \emptyset\} \\ &= \sup\{0 < \alpha \leq 1 : \cap_{i=0}^m L_\alpha(u_i) \neq \emptyset\}. \end{aligned}$$

- (ii) We have at least one optimal solution x^* for ((2.3),(2.4)), if and only if there exists an $\alpha_0 > 0$ such that

$$\cap_{i=0}^m L_{\alpha_0}(u_i) \neq \emptyset.$$

The above condition (ii) can be reduced to another type of condition by applying Ky Fan's theorem in [2] as follows:

Theorem K Let C be a compact and convex set in a topological linear space. Suppose that functions $f_i : C \rightarrow \mathbf{R}, i = 1, 2, \dots, n$, are lower semi-continuous and convex. Let $d \in \mathbf{R}$. Then the following (i) and (ii) are equivalent each other:

- (i) There exists an $x_0 \in C$ such that

$$f_i(x_0) \leq d$$

for $i = 1, 2, \dots, n$;

- (ii) for $c = (c_1, \dots, c_n)$ such that $c_i \geq 0, i = 1, 2, \dots, n$, and $\sum_{i=1}^n c_i = 1$, there exists a $y_c \in C$ satisfying

$$\sum_{i=1}^n c_i f_i(y_c) \leq d.$$

From the above theorem, Problem ((2.3),(2.4)) has an optimal solution x^* if and only if there exist $0 < \alpha_0 \leq 1$ and x_0 such that

$$u_i(a_i^T x_0) \geq \alpha_0$$

for $i = 0, 1, \dots, m$.

Theorem 2 Let $S = \cap_{i=0}^n S_i$ be non-empty and $u_i(\cdot)$ concave on $[C_i, D_i]$ for $i =$

$0, 1, \dots, m$. Then Problem ((2.3),(2.4)) has an optimal solution x^* , if and only if for some α_0 with $0 < \alpha_0 \leq 1$ and $c = (c_0, \dots, c_m) \in \mathbf{R}^{m+1}$ with $c_i \geq 0, i = 0, 1, \dots, m$, there exists a $y_c \in S$ such that

$$\sum_{i=0}^m c_i u_i(a_i^T y_c) \geq \alpha_0.$$

3 Quasiconvex Functions

In this section we suppose the quasiconvexity of membership functions and we show an extension of Ky Fan's theorem by applying the following lemma.

Lemma 1 Let C be a compact and convex set in a topological linear space E . Suppose that a set $A \subset C \times C$ satisfies the following conditions

(a) - (c):

(a) The set $\{x \in C : (x, y) \in A\}$ is closed for any $y \in C$;

(b) the set $\{y \in C : (x, y) \notin A\}$ is convex for any $x \in C$;

(c) for $x \in C$, the point $(x, x) \in A$.

Then there exists some $x_0 \in C$ such that $\{x_0\} \times C \subset A$.

The above Lemma can be proved by applying the following type of fixed points theorem for a class of set-valued mappings (e.g., Theorem 10.3.6 in [1]).

Theorem 3 Let E be a topological linear space and C a non-empty, compact and convex set in E . Let T be a mapping from C to the set of all subsets of C . Assume that the image $T(x)$ is non-empty and convex for

each $x \in C$. If for each $y \in C$, the inverse $T^{-1}(y) = \{x \in C : T(x) \ni y\}$ is open, then T has a fixed point in C , i.e., there exists an $x_0 \in C$ such that $x_0 \in T(x_0)$.

Proof of Lemma 1

Suppose that for any $x \in C$ there exists a $y \in C$ such that $(x, y) \notin A$. Denote a set-valued mapping T from C to the set of all subsets of C by $T(x) = \{y \in C : (x, y) \notin A\}$. The image $T(x) \subset C$ is non-empty and convex from Condition (b) for any $x \in C$. From Condition (a) the set $T^{-1}(y) = \{x \in C : (x, y) \notin A\}$ is open set in E . Then, by applying Theorem 3, T has a fixed point $x_0 \in C$, i.e., $x_0 \in T(x_0)$. It follows that $(x_0, x_0) \notin A$, which contradicts Condition (c). Thus the conclusion holds.

Q.E.D.

By utilizing the above lemma we think that the following results of an extension of Theorem K can be shown as the below outline of proof.

Extension of Theorem K(ETK)

- Let $f_i : C \rightarrow \mathbf{R}$ for $i = 1, \dots, n$, be lower semi-continuous and quasiconvex, where C is a compact and convex set in a topological linear space E and let $d \in \mathbf{R}$. Then the following (i) and (ii) are equivalent each other:

- (i) There exists an $x_0 \in C$ such that

$$f_i(x_0) \leq d$$

for $i = 1, 2, \dots, n$;

- (ii) for $c = (c_1, \dots, c_n)$ such that $c_i \geq 0$, $i = 1, 2, \dots, n$, and $\sum_{i=1}^n c_i = 1$, there

exists a $y_c \in C$ such that

$$\sum_{i=1}^n c_i f_i(y_c) \leq d.$$

In the similar way to the discussion of Chapter 6 in [7], we expect that we can prove the above extension.

4 Extensions of Minimax Theorems

[7] gives definitions of convexlike or concave-like functions, which play an important role in proving an extension of minimax theorems under that ETK holds.

Definition 3 Let C, D be two sets and F a mapping from $C \times D$ to \mathbf{R} . It is said that F is concavelike on D for $x \in C$ if for each $y_1, y_2 \in D$ and $0 < \lambda < 1$, there exists an $y_0 \in D$ such that $F(x, y_0) \geq \lambda F(x, y_1) + (1 - \lambda)F(x, y_2)$. It is said that F is convexlike on C for $y \in D$ if for each $x_1, x_2 \in C$ and $0 < \lambda < 1$, there exists an $x_0 \in C$ such that $F(x_0, y) \leq \lambda F(x_1, y) + (1 - \lambda)F(x_2, y)$.

In what follows we show an extension of minimax theorems concerning concavelike functions.

Extension of Minimax Theorems (EMT)

- Let C be a convex and compact set in a topological linear space and D an arbitrary non-empty set. A function $F : C \times D \rightarrow \mathbf{R}$ satisfies the following conditions (i) and (ii).

- (i) $F(\cdot, y)$ is lower semi-continuous and quasiconvex on C for $y \in D$;
 (ii) $F(x, \cdot)$ is concavelike on D for $x \in C$.

Then it follows that

$$\sup_{y \in D} \min_{x \in C} F(x, y) = \min_{x \in C} \sup_{y \in D} F(x, y).$$

Proof. From (i) and the compactness of C there exists $\min_{x \in C} F(x, y)$. Let $c = \sup_{y \in D} \min_{x \in C} F(x, y) < +\infty$. For any $x \in C$, $\{y_1, y_2, \dots, y_n\} \subset D$ and $\{\lambda_i \geq 0 : \sum_{i=1}^n \lambda_i = 1\}$, Condition (ii) means that there exists a $y_0 \in D$ such that $\sum_{i=1}^n \lambda_i F(x, y_i) \leq F(x, y_0)$. From (i) there exists an $x_0 \in C$ such that $F(x_0, y_0) = \min_x F(x, y_0) \leq c$ and also we have $\sum_{i=1}^n \lambda_i F(x, y_i) \leq c$ for any $x \in C$. By Condition (i) and ETk, there exists an $x_1 \in C$ such that $F(x_1, y_i) \leq c$ for any i . Then we get $\cap_{i=1}^n \{x \in C : F(x, y_i) \leq c\} \neq \emptyset$. From the compactness of C , we have $\cap_{y \in D} \{x \in C : F(x, y) \leq c\} \neq \emptyset$, which means that there exists an $x_2 \in C$ and any $y \in D$ such that $F(x_2, y) \leq c$, or $\min_x \sup_y F(x, y) \leq \sup_y \min_x F(x, y)$. Since $F(x, y) \geq \min_x F(x, y)$ for $y \in D$, we have $\sup_y F(x, y) \geq \sup_y \min_x F(x, y)$ and also $\min_x \sup_y F(x, y) \geq \sup_y \min_x F(x, y)$. Therefore $\sup_y \min_x F(x, y) = \min_x \sup_y F(x, y)$. If $\sup_{y \in D} \min_{x \in C} F(x, y) = \infty$, it can be seen that the conclusion holds.

Q.E.D.

The above theorem is an extension of Sion's minimax theorem and Tuy's one. In the following remark an example illustrates EMT.

Remark 3 (a) In [6] Sion assumes that F is upper semi-continuous and quasiconcave on

D under the condition that D is compact, in addition to the conditions of EMT. He gets the conclusion that

$$\min_{x \in C} \max_{y \in D} F(x, y) = \max_{y \in D} \min_{x \in C} F(x, y).$$

Thus EMT is an extension of Sion's theorem.

(b) Tuy [8] assumes that C and D are convex. He shows that the conclusion

$$\inf_{x \in C} \sup_{y \in D} F(x, y) = \sup_{y \in D} \inf_{x \in C} F(x, y)$$

under the condition that F is upper semi-continuous in y in addition to conditions of EMT.

(c) Let $F(x, y) = f(x)g(y)$ for $(x, y) \in [-n, n] \times (-1, 1)$, where $n \geq 1$ is integer, f denotes the largest integer which is less than $|x|$. Here

$$g(y) = y^2 + |y \sin \frac{\pi}{2y}|,$$

where $y \in (-1, 1)$. Then function F satisfies Conditions (i) and (ii) of EMT. Since $\min_x F(x, y) = 0$ and $\sup_y F(x, y) = 2f(x)$, It follows that the conclusion of EMT holds.

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